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The Role of Mathematics in Quantum Computing and Information Theory: Foundations, Applications and Future Direction

Sophie R. Beaumont

School of Engineering, Westhaven Institute of Technology, France

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Abstract

Mathematics provides the fundamental language, structural framework, and analytical machinery underlying quantum computing and quantum information theory. Unlike classical computation, which is based primarily on binary states and deterministic logical operations, quantum computing exploits quantum-mechanical phenomena such as superposition, entanglement, interference, and measurement. The mathematical description of these phenomena relies extensively on linear algebra, complex vector spaces, probability theory, operator theory, group theory, tensor products, and information-theoretic concepts. Quantum information theory further extends classical information theory by providing mathematical frameworks for describing quantum states, quantum channels, quantum measurements, quantum correlations, and the transmission and processing of quantum information.

This research paper examines the central role of mathematics in the development and operation of quantum computing and information theory. It discusses the mathematical representation of qubits using complex Hilbert spaces and explores the significance of vectors, matrices, unitary transformations, tensor products, eigenvalues, eigenvectors, and operators in quantum computation. The paper also examines the mathematical foundations of quantum algorithms, including the Quantum Fourier Transform, Grover's search algorithm, and Shor's factoring algorithm. In quantum information theory, concepts such as von Neumann entropy, quantum mutual information, quantum channels, and quantum error correction are considered. Particular attention is given to the relationship between mathematical abstraction and physical implementation and to the role of mathematics in addressing fundamental challenges such as decoherence, noise, scalability, and fault-tolerant quantum computation. The paper argues that mathematics is not merely a supporting tool for quantum computing but constitutes its essential conceptual and computational foundation. Future advances in quantum technologies are expected to depend increasingly on developments in mathematical optimization, algebra, geometry, probability, coding theory, topology, and information theory.

Keywords: Quantum Computing, Quantum Information Theory, Mathematics, Linear Algebra, Hilbert Space, Qubit, Quantum Algorithms, Quantum Entanglement, Quantum Error Correction, Quantum Information

1. Introduction

The development of quantum computing represents one of the most significant transformations in modern computational science. Classical computers process information using bits

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represented by states such as 0 and 1. Quantum computers instead employ quantum bits, or qubits, whose mathematical description permits combinations of basis states and whose collective behavior can exhibit entanglement and interference. These characteristics create fundamentally different computational possibilities.

Although quantum computing is rooted in quantum physics, mathematics provides the language through which quantum systems are represented, manipulated, analyzed, and verified. The mathematical foundations of quantum computing include linear algebra, complex analysis, probability theory, matrix theory, tensor algebra, group theory, functional analysis, optimization, coding theory, and information theory.

A single qubit can be represented mathematically as

[
|\psi\rangle=\alpha|0\rangle+\beta|1\rangle,
]

where (\alpha,\beta\in\mathbb{C}) and

[
|\alpha|^2+|\beta|^2=1.
]

The coefficients (\alpha) and (\beta) are complex probability amplitudes. This simple expression illustrates an important difference between classical and quantum information. A classical bit occupies one of two states, whereas a qubit is represented by a vector in a two-dimensional complex vector space.

As the number of qubits increases, the mathematical complexity grows rapidly. An (n)-qubit system is represented within a Hilbert space of dimension

[
2^n.
]

Consequently, even relatively small quantum systems require sophisticated mathematical techniques for their representation and analysis.

Quantum computing also relies heavily on matrix operations. Quantum gates are represented by unitary matrices that transform quantum states while preserving their normalization. For example, the Hadamard gate is

[
H=\frac{1}{\sqrt{2}}
\begin{pmatrix}
1&1\
1&-1
\end{pmatrix}.
]

Applying this transformation to (|0\rangle) produces

[
H|0\rangle=

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\frac{|0\rangle+|1\rangle}{\sqrt{2}}.
]
Thus, matrix algebra directly describes the operation of a quantum computer.
Information theory provides another major mathematical foundation. Classical information theory, developed principally through the work of Claude Shannon, introduced quantitative concepts such as entropy, information, communication capacity, and coding. Quantum information theory generalizes these concepts to systems whose information is encoded in quantum states.
The mathematical study of quantum information has implications extending beyond computation. Quantum communication, quantum cryptography, quantum sensing, quantum simulation, and quantum networks all depend on mathematical frameworks capable of describing quantum states and information.
The purpose of this paper is to examine the role of mathematics in quantum computing and information theory. It discusses the mathematical representation of quantum states, quantum operations, quantum algorithms, entanglement, information measures, quantum communication, error correction, and emerging mathematical directions. The paper argues that mathematics is not simply a technical tool used to implement quantum technologies; it is the underlying framework through which the concepts of quantum information and computation become precise and analyzable.

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2. Mathematical Foundations of Quantum Computing

Quantum computing depends on several interconnected mathematical disciplines. Linear algebra is perhaps the most fundamental because quantum states are represented by vectors and quantum operations by matrices and linear operators.
Complex numbers are essential because quantum amplitudes can possess both magnitude and phase. If
[
z=a+bi,
]
then its magnitude is
[
|z|=\sqrt{a^2+b^2}.
]
Quantum probabilities are obtained from the squared magnitude of probability amplitudes.
Vector spaces provide the mathematical setting for representing quantum states. A qubit belongs to a two-dimensional complex vector space. More generally, an (n)-qubit system belongs to a (2^n)-dimensional complex Hilbert space.
Matrices represent quantum transformations. A quantum operation must preserve the total probability, which is expressed mathematically through unitary transformations.
A matrix (U) is unitary if

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[
U^\dagger U=I,
]
where (U^\dagger) is the conjugate transpose of (U).
These mathematical conditions ensure that quantum evolution preserves the norm of the state vector.

---

3. Qubits and Complex Vector Spaces

The qubit is the basic unit of quantum information.
A classical bit has two possible states:
[
0,\quad 1.
]
A qubit is represented by
[
|\psi\rangle=\alpha|0\rangle+\beta|1\rangle.
]
The basis states are represented by the vectors
[
|0\rangle=
\begin{pmatrix}
1\
0
\end{pmatrix},
\qquad
|1\rangle=
\begin{pmatrix}
0\
1
\end{pmatrix}.
]
Therefore,
[
|\psi\rangle=
\begin{pmatrix}
\alpha\
\beta
\end{pmatrix}.
]
The normalization condition is

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[
|\alpha|^2+|\beta|^2=1.
]
When a measurement is made in the computational basis, the probability of obtaining 0 is
[
P(0)=|\alpha|^2,
]
while the probability of obtaining 1 is
[
P(1)=|\beta|^2.
]
This mathematical representation provides the basis for quantum information processing.

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4. Hilbert Spaces and Quantum States

A Hilbert space is a complete inner-product vector space. In quantum theory, it provides the mathematical environment in which states and operators are defined.
For a single qubit, the Hilbert space is
[
\mathcal{H}=\mathbb{C}^2.
]
For two qubits,
[
\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2,
]
which has dimension four.
For (n) qubits,
[
\mathcal{H}=(\mathbb{C}^2)^{\otimes n},
]
with dimension
[
2^n.
]
The exponential growth of Hilbert-space dimension is both a source of quantum computational power and a major challenge for classical simulation.
Tensor products are therefore central to quantum computing. They allow individual quantum systems to be combined into larger composite systems.

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5. Quantum Gates and Matrix Algebra

Quantum gates are mathematical transformations acting on quantum states.
The Pauli matrices are fundamental single-qubit operators:

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$$
X=
\begin{pmatrix}
0&1\\
1&0
\end{pmatrix},
$$

$$
Y=
\begin{pmatrix}
0&-i\\
i&0
\end{pmatrix},
$$

and

$$
Z=
\begin{pmatrix}
1&0\\
0&-1
\end{pmatrix}.
$$

The (X) gate is analogous to a classical bit flip because

$$
X|0\rangle=|1\rangle
$$

and

$$
X|1\rangle=|0\rangle.
$$

The Hadamard gate produces superposition:

$$
H|0\rangle=
\frac{|0\rangle+|1\rangle}{\sqrt2}.
$$

Quantum circuits are constructed by applying sequences of such transformations.

The mathematical composition of gates corresponds to matrix multiplication. If gates (U) and (V) are applied sequentially, the combined operation is

$$
VU.
$$

Thus, linear algebra directly determines the evolution of quantum computations.

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6. Measurement and Probability Theory

Quantum measurement introduces probability into the mathematical description of quantum systems.

A measurement does not generally reveal all components of a quantum state simultaneously. Instead, possible measurement outcomes are associated with probabilities determined by the state and measurement operators.

For a projective measurement with projectors (P_i), the probability of outcome (i) can be expressed as

[
p_i=\langle\psi|P_i|\psi\rangle.
]

The mathematical framework therefore combines linear algebra with probability theory.

For a density matrix (\rho), the probability of a measurement outcome represented by (M_i) is

[
p_i=\operatorname{Tr}(M_i\rho).
]

This formalism is essential when describing mixed states, noisy systems, and quantum channels.

7. Quantum Entanglement and Tensor Algebra

Entanglement is one of the most distinctive features of quantum information.

Consider the two-qubit state

[
|\Phi^+\rangle=
\frac{|00\rangle+|11\rangle}{\sqrt2}.
]

This state cannot be expressed as a simple product of two independent single-qubit states. It therefore represents an entangled state.

Mathematically, entanglement arises naturally from tensor-product structures.

Entanglement has important consequences for quantum communication and computation. It enables correlations that cannot be reproduced by classical probabilistic models under the same assumptions.

The mathematical study of entanglement involves tensor algebra, matrix decompositions, operator theory, convex geometry, and information theory.

Measures of entanglement can quantify how strongly two quantum systems are correlated. Such measures are important for understanding quantum resources and designing quantum protocols.

8. Quantum Information Theory

Quantum information theory extends classical information theory to quantum systems.

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In classical information theory, entropy quantifies uncertainty. For a discrete random variable (X) with probabilities (p(x)), Shannon entropy is

[
H(X)=-\sum_x p(x)\log p(x).
]

For a quantum state represented by density matrix (\rho), the corresponding von Neumann entropy is

[
S(\rho)=-\operatorname{Tr}(\rho\log\rho).
]

The eigenvalues of (\rho) play a central role in calculating this quantity.

Quantum entropy provides a mathematical measure of uncertainty and information content in a quantum state.

It also plays an important role in quantum communication, thermodynamics, cryptography, and the study of entanglement.

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9. Quantum Mutual Information

Mutual information measures the degree of dependence between two systems.

For classical variables (X) and (Y),

[
I(X;Y)=H(X)+H(Y)-H(X,Y).
]

For quantum systems (A) and (B), quantum mutual information can be written as

[
I(A:B)=S(\rho_A)+S(\rho_B)-S(\rho_{AB}),
]

where (\rho_A) and (\rho_B) are reduced density matrices and (\rho_{AB}) is the joint state.

This quantity provides information about correlations between quantum systems.

The mathematical structure of quantum mutual information is important for studying entanglement, communication, quantum networks, and information transfer.

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10. Quantum Channels

Quantum communication requires mathematical descriptions of how quantum states change when transmitted through physical systems.

A quantum channel can be represented by a completely positive trace-preserving map

[
\mathcal{E}:\rho\rightarrow\mathcal{E}(\rho).
]

Such maps describe noise, decoherence, transmission, and other transformations.

For example, a noisy channel may transform a pure quantum state into a mixed state.

The mathematical study of quantum channels involves operator theory, matrix analysis, probability, and information theory.

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Channel capacity is another important concept. It concerns the maximum rate at which information can be reliably transmitted through a communication system under specified assumptions.

Quantum channel theory therefore provides the mathematical foundation for quantum communication technologies.

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11. Quantum Fourier Transform

The Quantum Fourier Transform (QFT) is a fundamental component of several quantum algorithms.

For a computational basis state (|x\rangle) of dimension (N), the QFT can be expressed as

[
|x\rangle
\rightarrow
\frac{1}{\sqrt{N}}
\sum_{y=0}^{N-1}
e^{2\pi ixy/N}|y\rangle.
]

The transformation relies heavily on complex numbers, roots of unity, linear algebra, and Fourier analysis.

The QFT illustrates how classical mathematical concepts can become powerful computational operations in quantum systems.

Its significance extends beyond a single algorithm. Fourier analysis is central to signal processing, harmonic analysis, differential equations, and number theory, making the QFT an important intersection between mathematics and quantum computation.

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12. Shor's Algorithm and Number Theory

Shor's algorithm demonstrates the connection between quantum computing and number theory. The algorithm provides a quantum approach to integer factorization that is substantially more efficient asymptotically than the best known general-purpose classical factoring methods under standard complexity assumptions.

A central step involves period finding.

Suppose a periodic function satisfies

[
f(x+r)=f(x).
]

The Quantum Fourier Transform can be used to extract information about the period (r).

Number theory then connects the discovered period to the factorization problem.

The mathematical structure of Shor's algorithm therefore combines:

- modular arithmetic,
- number theory,
- group theory,
- Fourier analysis,

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- probability,
- linear algebra, and
- quantum circuit theory.

This demonstrates that quantum algorithms can emerge directly from deep mathematical structures.

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13. Grover's Algorithm and Mathematical Search

Grover's algorithm provides a quantum approach to searching an unstructured space.

For a search space containing (N) possibilities, classical exhaustive search may require an average number of queries proportional to (N), whereas Grover's algorithm requires approximately

[
O(\sqrt{N})
]

queries.

The algorithm relies on amplitude amplification.

Mathematically, Grover's procedure can be interpreted as repeated rotations in a two-dimensional subspace.

This geometric interpretation demonstrates the role of linear algebra and geometry in quantum algorithm design.

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14. Quantum Error Correction

Quantum systems are highly sensitive to environmental noise. Decoherence can destroy quantum information before computation is completed.

Quantum error correction addresses this problem mathematically by encoding information across multiple physical qubits.

Unlike classical error correction, quantum error correction must deal with errors affecting amplitudes and phases.

A general quantum error-correcting code can be represented through a mapping

[
|\psi\rangle
\rightarrow
|\psi_L\rangle,
]

where the logical state (|\psi_L\rangle) is encoded across multiple physical qubits.

Mathematical coding theory provides concepts for understanding how errors can be detected and corrected.

Quantum error correction is also closely connected to linear algebra, stabilizer theory, group theory, and increasingly topology.

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15. Topology and Quantum Computing

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Topology has emerged as an important mathematical area in quantum information and quantum computing.
Topological concepts can be used to describe robust structures that are less sensitive to local disturbances.
Topological quantum computing explores computational schemes in which information is encoded using topological properties of quantum systems.
Mathematical tools involved include:
[
\text{topological spaces, knots, braids, groups, and category-theoretic structures}.
]
Braiding operations, for example, can provide mathematical mechanisms for representing quantum gates in certain theoretical architectures.
Although practical realization remains an active research challenge, the mathematical framework demonstrates the increasing intersection of abstract mathematics and quantum technology.

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16. Group Theory and Quantum Algorithms

Group theory provides another important mathematical foundation.
Quantum systems often possess symmetries that can be represented using groups and their representations.
The study of group representations can help determine how quantum states transform under symmetry operations.
Several quantum algorithms exploit algebraic structures. Period-finding and hidden-subgroup problems are prominent examples.
The hidden subgroup framework connects quantum computing with abstract algebra and has influenced the development of algorithms for problems involving groups.
This demonstrates that quantum computational advantage can be closely connected to mathematical structure rather than merely computational speed.

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17. Mathematical Complexity Theory

Quantum computing also requires a mathematical theory of computational complexity.
Classical complexity theory studies the resources required to solve computational problems.
Quantum complexity theory introduces quantum computational models and complexity classes.
One of the most important classes is BQP, representing problems that can be solved efficiently with bounded error by quantum computers under a standard quantum circuit model.
The comparison between classical and quantum complexity classes remains an important theoretical research area.
Questions concerning relationships between classes such as P, BPP, BQP, NP, and others involve mathematical logic, theoretical computer science, probability, and algebra.
Complexity theory helps establish which problems may benefit from quantum computation and which may not.

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18. Optimization and Quantum Computing

Optimization is another area where mathematics connects strongly with quantum computing.
Many practical problems can be represented as optimization tasks:

[
\min_x f(x).
]

Quantum algorithms and quantum-inspired approaches are being investigated for combinatorial optimization, scheduling, portfolio optimization, routing, and other problems.
Quantum approximate optimization approaches seek to construct quantum states that encode solutions to optimization problems.
The mathematical analysis of these methods involves probability distributions, linear algebra, graph theory, variational calculus, and optimization theory.
The practical advantage of quantum optimization remains an active area of research, but its mathematical foundations are already substantial.

19. Quantum Machine Learning

Quantum computing and machine learning increasingly intersect through quantum machine learning.
Classical machine learning depends heavily on mathematical tools such as linear algebra, probability, optimization, statistics, and information theory. Quantum machine learning extends some of these concepts into quantum computational settings.
Quantum states can encode high-dimensional information, while quantum circuits can perform transformations on these representations.
A simplified parameterized quantum circuit can be written as

[
|\psi(\theta)\rangle=U(\theta)|\psi_0\rangle,
]

where (\theta) represents trainable parameters.
An optimization process can then attempt to find

[
\theta^*=\arg\min_\theta L(\theta),
]

where (L) is an appropriate loss function.
The field remains developing, and determining when quantum machine learning provides meaningful practical advantages is an important research question.

20. The Role of Probability and Statistics

Probability theory is fundamental to quantum mechanics because measurement outcomes are probabilistic.
Statistical methods are equally important because real quantum systems are affected by experimental uncertainty and noise.

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Researchers must estimate parameters from experimental observations, characterize quantum states, and determine whether measured outcomes agree with theoretical predictions.
Statistical inference can therefore be integrated with quantum-state reconstruction, quantum process characterization, and experimental calibration.
Bayesian methods may also be used to represent uncertainty about quantum-system parameters.
The combination of probability, statistics, and quantum theory provides a mathematical foundation for extracting reliable information from noisy quantum experiments.

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21. Mathematical Challenges in Quantum Computing

Despite rapid progress, quantum computing presents several difficult mathematical challenges.
One major challenge is the exponential growth of quantum state spaces. An (n)-qubit system requires a state vector with (2^n) complex amplitudes in the general case.
Another challenge concerns optimization. Quantum algorithms often require the adjustment of many parameters, and optimization landscapes may contain flat regions, local structures, or noise-induced difficulties.
Noise presents another mathematical challenge. Real quantum systems cannot be treated as perfectly isolated systems. Mathematical models must therefore account for decoherence and imperfect operations.
Verification is also difficult. Demonstrating that a large quantum computation produced a correct result can itself require sophisticated mathematical and computational techniques.
These challenges make mathematical research essential to the development of scalable quantum computing.

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22. Future Directions

The future development of quantum computing is likely to depend heavily on mathematical innovation.
One major direction is fault-tolerant quantum computation. Improved mathematical codes and error-correction strategies will be essential for protecting quantum information.
Another direction involves quantum algorithms based on advanced algebraic and geometric structures. Greater understanding of mathematical symmetries may lead to new algorithms.
Quantum information theory is also expected to contribute to quantum communication networks and distributed quantum computation.
Mathematical optimization will remain important for circuit design, parameter estimation, error correction, and hardware control.
Topology, category theory, operator algebras, and mathematical physics may provide additional theoretical frameworks for understanding emerging quantum architectures.
There is also significant potential for AI-assisted mathematical research in quantum computing.
Machine learning systems may help discover quantum circuits, optimize algorithms, identify mathematical patterns, and search large spaces of possible error-correcting codes.

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23. Conclusion

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Mathematics constitutes the fundamental language of quantum computing and information theory. Concepts that may initially appear exclusively physical or technological—such as qubits, quantum gates, entanglement, quantum channels, and quantum algorithms—are expressed precisely through mathematical structures.

Linear algebra provides the representation of quantum states and operations. Complex numbers describe probability amplitudes and phase. Tensor products describe composite systems. Probability theory describes measurement. Operator theory provides a framework for quantum transformations. Information theory quantifies uncertainty and information. Group theory and number theory support important quantum algorithms. Coding theory and topology contribute to quantum error correction. Optimization and statistics assist in developing and implementing quantum systems.

Quantum algorithms such as Shor's factoring algorithm, Grover's search algorithm, and the Quantum Fourier Transform demonstrate how mathematical ideas can be transformed into computational procedures with potentially significant advantages.

The mathematical study of quantum information also provides a deeper understanding of communication, entropy, correlations, noise, and information processing. As quantum technologies mature, mathematics will become increasingly important not only for theoretical development but also for algorithm design, error correction, system verification, and optimization.

The relationship is ultimately reciprocal. Mathematics enables quantum computing, while quantum computing generates new mathematical problems involving complexity, information, geometry, probability, algebra, topology, and optimization. Consequently, the future of quantum computing is likely to be accompanied by substantial developments across multiple mathematical disciplines.

Quantum computing should therefore not be understood solely as a new type of computer hardware. It represents a broader mathematical and scientific paradigm in which abstract mathematical structures become directly connected to physical information processing. The continued integration of mathematics, quantum physics, information theory, and computer science will be essential for realizing the full potential of quantum technologies.

References

1. Nielsen, M. A., & Chuang, I. L. (2010). *Quantum Computation and Quantum Information*. Cambridge University Press.
2. Wilde, M. M. (2017). *Quantum Information Theory*. Cambridge University Press.
3. Preskill, J. (2018). *Lecture Notes for Physics 219: Quantum Computation*. California Institute of Technology.
4. Schumacher, B. (1995). Quantum coding. *Physical Review A, 51*, 2738–2747.
5. Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal, 27*, 379–423, 623–656.
6. von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*. Princeton University Press.
7. Dirac, P. A. M. (1981). *The Principles of Quantum Mechanics*. Oxford University Press.

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8. Shor, P. W. (1994). Algorithms for quantum computation: Discrete logarithms and factoring. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, 124–134.
9. Grover, L. K. (1996). A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, 212–219.
10. Bennett, C. H., & Brassard, G. (1984). Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing*, 175–179.
11. Bennett, C. H., DiVincenzo, D. P., Smolin, J. A., & Wootters, W. K. (1996). Mixed-state entanglement and quantum error correction. *Physical Review A, 54*, 3824–3851.
12. Calderbank, A. R., & Shor, P. W. (1996). Good quantum error-correcting codes exist. *Physical Review A, 54*, 1098–1105.
13. Steane, A. M. (1996). Error correcting codes in quantum theory. *Physical Review Letters, 77*, 793–797.
14. Deutsch, D. (1985). Quantum theory, the Church–Turing principle and the universal quantum computer. *Proceedings of the Royal Society A, 400*, 97–117.
15. Feynman, R. P. (1982). Simulating physics with computers. *International Journal of Theoretical Physics, 21*, 467–488.
16. Nielsen, M. A. (2002). A geometric approach to quantum circuit lower bounds. *Quantum Information & Computation, 2*, 341–360.
17. Watrous, J. (2018). *The Theory of Quantum Information*. Cambridge University Press.
18. Aaronson, S. (2013). *Quantum Computing Since Democritus*. Cambridge University Press.
19. Wilde, M. M. (2023). *Quantum Information Theory*. Cambridge University Press.
20. Horodecki, R., Horodecki, P., Horodecki, M., & Horodecki, K. (2009). Quantum entanglement. *Reviews of Modern Physics, 81*, 865–942.